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COMPLETE INTEGRABILITY OF HAMILTONIAN SYSTEMS AND DIFFERENTIAL GALOIS GROUPS

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The subject of the lecture is a recent new approach of the old problem of integrability of systems of classical mechanics based upon *differential Galois theory and algebraic groups.* (The lecture is mainly based upon a joint work with *Juan Morales* from Barcelona.)

Roughly speaking an hamiltonian system with n degrees of freedom is integrable if there exists sufficiently many first integrals (integrals of motion) such that its integration can be reduced to quadratures. Here we are mainly interested in *completely integrable systems* in the classical Liouville sense (i.e. there must exists n first integrals generically independent and in involution). The investigation of such systems was a central subject during the last century and stimulated the appearance of the theory of Lie groups. Single out integrable systems among all the Hamiltonian systems remain a big open question. H. Poincaré proved that the existence of global integrals of motion is exceptional, a fortiori completely integrable systems are even more exceptional and only a small number of such systems are known. Recently the situation was perfectly stated by Perelomov (1990): *To find a general criterion for complete integrability seems at present a hopeless task.* In this lecture we will present such a criterion. It will sound quite abstract, but it is (surprisingly even for me...) easy to apply to actual problems: it gives new elegant solutions of some classical problems (as Lagrange top) and allows us to solve very easily a lot of open problems.

1. Ordinary differential equations, first integrals, linearized (variational) equations.
2. Hamiltonian systems (the symplectic formulation). Completely integrable systems.
3. A first (weak) formulation of our theorem: if an hamiltonian system is completely integrable, then the Lie algebra of the Zariski closure in the complex linear symplectic group $Sp(2n; C)$ of the monodromy group of a variational equation along a solution (in complexified time) is *abelian.*
4. Basics about differential Galois theory
5. Our main theorem: if an hamiltonian system is completely integrable,

then the Lie algebra of the differential Galois group of the variational equation along any solution is *abelian.*

6. Applications.

Recipe for applications. Examples of solutions of classical open problems:

Collinear three body problems with homogeneous potentials, Bianchi IX Cosmological model, spring pendulum... (by J.Morales and J.P. Ramis), Collinear four bodies problems with potential $1/r^2$ (solution of the first open problem after Jacobi and Moser-Calogero results, by Emmanuelle Juillard-Tosel 1999), non integrability of the planar Newton three bodies problem near the equilateral Lagrange solution by Delphine Boucher and Alexei Tsygvintsev 1999), Henon-Heiles systems...